lectures in Topological Spaces-Department of Mathematics -Fourth stage

Syllabus

1- ***Definitions and (Examples) of a Topological Space.***

2- ***Types of Topological Spaces.***

3- ***Closed subsets of a topological space.*** **4-** ***Neighborhoods.***

5- ***Closure of a Set.*** **6-** ***Topologies Induced by Functions.***

7- ***Interior of a Set, Exterior of a Set, Boundary of a Set and Cluster Points.***

8- ***Dense Subset of the Space. 9- Dense Subset of the Space.***

10- ***Continuous Functions.***

11- ***Open and Closed mappings***

12- ***Homeomorphisms.***
13- ***Topological spaces and Hereditary Property.***

14- ***Compactness in Topological Spaces.***

15- ***Connectedness in Topological Spaces.***

16- ***Separation Axioms and study relationships between them.***

Topological Spaces

The first lecture

Definition:

Let X be a non-empty set. Then the collection T of sub sets of X is called Topology for X if T satisfies the following axioms:-

1- X and $\emptyset \in T$.

2- If A_1 and A_2 are any two sets in T. then $A_1 \cap A_2 \in T$.

3- If $\{A_\alpha : \alpha \in \Delta\}$ be an arbitrary collection of sets in T then $\cup \{A_\alpha : \alpha \in \Delta\}$ is in T.

Remark:

If T is topology on X. Then (X, T) is called Top-space.

Remark:

In a topological space $(X.T)$. The members of T are called open sets.

So: in a topological space $(X.T)$:-

1- $\emptyset$, X are open sets

2- The intersection of finite collection of open sets is open.

3- Arbitrary (in finite) union of open sets is open.

Examples:

Let $X = \{a, b, c\}$ consider the following collection of subset of X:

$T_1 = \{\emptyset, X, \{a\}\}$ **and** $T_2 = \{\emptyset, X, \{a\}, \{a, c\}\}$.

It's clear that each one of above collections or families are topology or *X*

***T_3* = {Ø, X, {a}, {c}} not topology on X because {a} ∈ T_3 and {c} ∈ T_3 But {a} ∪ {c} = {a, c} ∉ T_3.**

Some types of topological space

First: **Let *X* ≠ Ø. The collection T_i = {Ø, X} is topology and it known indiscrete topology.**

The pair: *(X, T_i)* is called Top-sp

Second: **X≠Ø and T_d is collection of all possible subsets of X. then T_d is topology for X. (i. e) T_d = {power (X) = {P(X)}**

Third: **Let X ≠Ø and T* = {U: X-U is finite} ∪ {Ø}**

(i.e) T* consist of Ø and all non-empty subsets of X whose complement are finite.

Then (X , T_c) is called co-finite Top.

Fourth: **Let X ≠ Ø and T^c = {U: X-U countable} ∪ {Ø}**

Then (X, T^c) is called co-countable Topological space..

Fifth: **Let *X* = *R* be all a real numbers and Let T_u be a family consisting of Ø and all non-empty subsets G of R which have the following property:-**

{∀ x ∈ G} open interval I_x such that X ∈ I_x ⊆ G, Then (X, T_u) is called usual Topological space.

Comparison of Topologies-The second lecture

Definition:

Let T_1 and T_2 be any two topologies for a set $X \neq Ø$:-

1) **If every open set in T_1 is open set in T_2 then we write $T_1 \subset T_2$ and say :**

   T_1 is coarser or weaker or smaller than T_2 or T_2 is finer or stronger or longer than T_1.

2) **If either $T_1 \subset T_2$ or $T_2 \subset T_1$ we say that T_1 and T_2 comparable otherwise we say not comparable.**

Definition:

Let (X, T) be a topology space a subset F of X is said to be closed if the complement $F^c \in T$

Intersection and union of open and closed set

Theorem:

1- **The intersection of a finite collection of open sets is open .**
2- **The intersection of finite collection of open sets not necessarily open set.**
3- **The union of in finite the collection of open sets is open.**

Theorem:

1- **The union of finite collection of closed sets is closed.**

2- The union of in finite collection of closed sets not necessarily closed set.

3- The intersection of in finite collection of closed sets is closed.

Definition:

A topological space (X,T) is called door space . If every subset of X is either open or closed.

Definition:

Let (X, T) be a topological space and let $x \in X$. Then a subset N of X is said to be:-

T-neighborhood or neighborhood of x if there exists open set G such that $x \in G \subseteq N$.

Definition:

The set of all neighborhoods of a point $x \in X$ is called the neighborhood system of x and denoted by N_x.

Definition:

Let (X, T) be a topological space. Let $x \in X$ and let N_x be the T – neighborhood system of X. Then the sub family β_X of N_x is called local base of x if for each $N \in N_x \ \exists\ B \subseteq B_x$ such that $X \in B \subseteq N$.

Definition:

Let (X, T) be a topology space. a sub family β of T is said to be form a base for T if for each open set G and each $x \in G \ \exists$ a member B in β such that $x \in B \subseteq G$

Limit points and closure of sets- The third lecture

Definition:

Let (X, T) be a topology space and let A ⊆ X A point x ∈ X is called adherent point or contact point of A if every open set containing X. Contains at least one point of A

Definition:

A point x ∈ X is called a limit point or accumulation point of A or a cluster point of A if and only if every open set containing x contains at least on point of A other than x .

Remark:

The set of all limit points of A is called the derived set of A and will denoted by À or D_r (A)

Theorem:

Let (X, T) be a topology space and let A ⊆ X. Then A is closed if and only if À ⊆ A . or D(A) ⊆ A .

Theorem:

Let (X, T) be a topological space and let A and B be any subset of X then:

1. **$\emptyset^{\backslash}$ = Ø or D(Ø) = Ø**
2. **If A ⊆ B ⇒ D(A) ⊆ D(B)**

3- $D(A \cap B) \subseteq D(A) \cap D(B)$

4- $D(A \cup B) = D(A) \cup D(B)$

Definition:

Let (X,T) be a topological space and let $A \subseteq X$, then the intersection of all closed sets of A is called the closure of A and denoted by $\bar{A}$ or C/(A).

Theorem:

Let (X, T) be a topological space and let $A \subseteq X$. Then $\bar{A}$ is the smallest closed of A {contains A}.

Theorem:

Let (X, T) be topological space and let $A \subseteq X$ then A is closed if and only if $\bar{A} = A$.

Theorem:

Let (X, T) be a topological space and let A and B be a subsets of X then:-

1- $\overline{\emptyset} = \emptyset$ and $\overline{X} = X$ and $\overline{\overline{A}} = \overline{A}$

2- If $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$

3- $(\overline{A \cap B}) \subseteq (\overline{A} \cap \overline{B})$

4- $(\overline{A \cup B}) \subseteq (\overline{A} \cup \overline{B})$.

Interior, Exterior and Boundary of sets- The fourth lecture

Definition:

Let (X, T) be a topological space and let $A \subseteq X$, a point $x \in A$ is said to be a interior point of A if and only if A is a neighborhood of x.

Remark: **the sets of all interior points of A is called the interior of A and denoted by int (A) or A^0.**

Theorem: **Let (X, T) be a topological space and let $A \subseteq X$, then:-**

1- A^0 is the largest open subsets contained in A.

2- A is open if and only if and only if $A^0 = A$ or int (A) = A.

Theorem: **Let (X,T) be a topological space and let A and B be any subsets of X , then :- 1- $\emptyset^0 = \emptyset$, $X^0 = X$ and $(A^0)^0 = A^0$. 2-If $A \subseteq$ \b $\Rightarrow A^0 \subseteq B^0$**

3- $(A \cap B)^0 = A^0 \cap B^0$.

Definition: **Let (X, T) be a topological space and let A ⊆ X a point x ∈ X is called an exterior point of A if and only if it is an interior point of A^c.**

Remark: **The set of all exterior points of A is called the exterior of A and denoted by ext (A).**

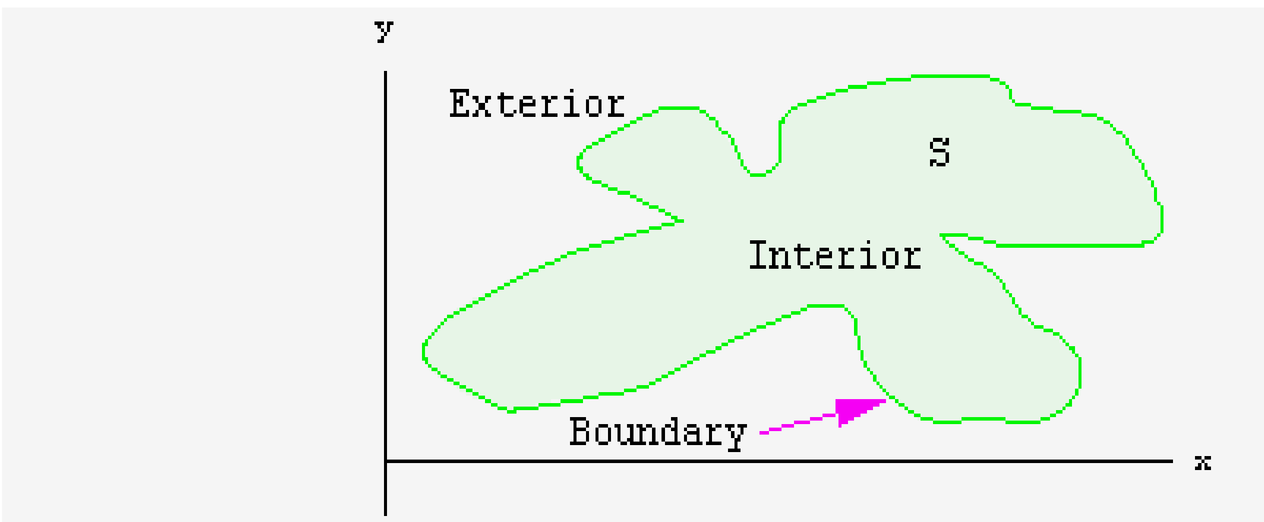
Definition: **Let (X, T) be a topological space and let A ⊆ X. A point x ∈ X is called boundary point or frontier point of A if and only if :-**

Let (X, T) be a topological space and let A ⊆ X. A point x ∈ X is called boundary point or frontier point of A if and only if :-

Every open set containing x intersects both A and A^c or A and cl (A).

Remark:

The set of all boundary point is called the boundary of A written as bd (A) or Fr(A)



Definition: **Let (X, T) be a topological space and let $A \subseteq X$, then A is said to be:**

1- Everywhere dense if $\bar{A} = X$. 2- Nowhere dense if $ext(A) = X$.

3- Dense in itself if $\bar{A} \subseteq A$ (i.e.) every limit point of A is in A.

4- Dense relative to another set B , if $B \subseteq \bar{A}$.

Definition: **A topological space (X, T) is said to be separable if and only if there exists a countable dense subset A of X.**

Definition

Let (X, T) be a topological space and let $Y \subseteq X$, then The collection $T_y = \{G \cap Y : G \in T\}$ is topology on X.

Sub-Spaces on Topological Space- The fifth lecture

Introduction:

It is always possible to construct new topologies from the given ones. The simplest one is the relativized Topology.

If (X, T) is topological space and $Y \subset X$, then Y can inherit a topology from X; as shown in the following result.

Definition:

If (X, T) is topological space and $Y \subset X$, then, the collection $T_Y = \{G \cap Y : G \in T\}$ is a topology on Y.

Proof: **We observe that T_Y satisfies the following properties:**

$\emptyset \in T \text{ and } \emptyset \cap Y = \emptyset \Rightarrow \emptyset \in T_Y;$

$X \in T \text{ and } X \cap Y = Y \Rightarrow Y \in T_Y;$

Let $\{H_\alpha : \alpha \in \Delta\}$ be any family of sets in T_Y.

Then, for each $\alpha \in \Delta \; \exists$ a set $G_\alpha \in T$ such that $H_\alpha = G_\alpha \cap Y$.

$\cup \{H_\alpha : \alpha \in \Delta\} = \cup \{G_\alpha \cap Y : \alpha \in \Delta\}$

$$= [\cup \{G_\alpha \cap Y : \alpha \in \Lambda\}] \cap Y \in T_Y, \text{since } \cup \{G_\alpha : \alpha \in \Lambda\} \in T;$$

Let H_1 and H_2 be any two sets in T_Y.

Then $H_1 = G_1 \cap Y$ and $H_2 = G_2 \cap Y$ for some G_1, G_2 in T.

$H_1 \cap H_2 = (G_1 \cap Y) \cap (G_2 \cap Y)$

$$= (G_1 \cap G_2) \cap Y \in T_Y, \text{since } G_1 \cap G_2 \in T.$$

Hence, T_Y is a topology for Y.

Remark:

This topology T_Y is the relativized or inherited topology on Y. Also (Y, T_Y) is called the sub-space of (X, T_Y).

Example

Let $X = \{a, b, c, d, e\}$ and let $T = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$ be a topology on X.

Let $Y = \{a, d, e\}$.

Then, we have

$X \cap Y = Y$; $\emptyset \cap Y = \emptyset$; $\{a\} \cap Y = \{a\}$; $\{c, d\} \cap Y = \{d\}$;

$\{a, c, d\} \cap Y = \{a, d\}$**and** $\{b, c, d, e\} \cap Y = \{d, e\}$.

T relativized topology on Y is given by

$T_Y = \{Y, \emptyset, \{a\}, \{d\}, \{a, d\}, \{d, e\}\}$.

Example:

Consider the usual topological space (R, u). Let N be the set of all natural numbers. Then, the relativized topology u_N on N is the discrete topology.

Proof: **For an arbitrary $n \in N$, we have**

$\{n\} = \left[n - \frac{1}{2}, n + \frac{1}{2}\right] \cap N \in u_N$ **since** $\left[n - \frac{1}{2}, n + \frac{1}{2}\right] \in u$.

Thus, each singleton subset of N is u_N- open.

Now, if A is any subset of N, then it can be expressed as the union of singleton subsets of N, each one of which is u_N- open.

And, the arbitrary union of sets being open, it following that A is u_N- open. Thus, every subset of N is u_N- open. Hence, the relativized topology for N is the discrete topology.

Remark:

If (Y, T_Y) is a sub-space of the space (X, T_Y), then a set open in X is not necessarily open in Y.

Sub-Spaces on Topological Space-Continued- The sixth lecture

Theorem:

Let (Y, T_Y) be a sub-space of a topological space (X, T), then, a subset A of Y is T_Y-closed if and only if $A = F \cap Y$, for some T-closed subset F of X.

Proof:

A is T_Y- closed

$\Leftrightarrow (Y - A)$ **is** T_Y**-open**

$\Leftrightarrow (Y - A) = G \cap Y$ **for some** $G \in T$

$\Leftrightarrow A = Y - (G \cap Y) = (Y - G) \cup (Y - Y)$**[by De-Morgan's law]**

$\Leftrightarrow A = (Y - G) = Y \cap G^c$ **is T-closed**

$\Leftrightarrow A = Y \cap F$**,where** $F = G^c$**is T-closed.**

Theorem:

Let(X, T), (Y, T^*), and (Z, T^{**}) be three topological spaces such that (Y, T^*) is a subspace of (X, T) and (Z, T^{**}) be a subspace of (X, T^*). Then, (Z, T^{**}) is a subspace of (X, T).

Proof:

Clearly, Y⊂X and Z⊂Y, so, Z⊂X

In order to prove that (Z, T^{**}) is a sub-space of (X, T), we must show that the T-relativized topology on Z is T^{**} i.e. T.= T^{**}.

Let $E \in T^{**}$. Then,

$E = H \cap Z$ **for some** $H \in T^*$ **[(Z, T^{**}) is a sub space of (Y, T^*)]**

$= (G \cap Y) \cap Z$ **for some** $G \in T$ **[(Y, T^*) is a sub space of (X, T)]**

$= G \cap (Y \cap Z) = G \cap Z$ **[Z⊂Y]**

Thus, $E = G \cap Z$ for some $G \in T$ and therefore, $E \in T_Z$

so, $E \in T^{**} \Rightarrow E \in T_Z$ i.e. $T^{**} \subseteq T_Z$.

Again, let $W \in T_Z$. Then, $W = V \cap Z$ for some $V \in T$.

But, $V \in T \Rightarrow V \cap Y \in T^{*}$ $\quad$ [(Y, T^{*}) is a sub space of (X, T)]

$\Rightarrow (V \cap Y) \cap Z \in T^{**}$ $\quad$ [(Z, T^{**}) is a sub space of (Y, T^{*})]

$\Rightarrow V \cap (Y \cap Z) = V \cap Z \in T^{**}$

$\Rightarrow W \in T^{**}$

Thus, $W \in T_Z \Rightarrow W \in T^{**}$ and therefore, $T_Z \subseteq T^{**}$.

Hence, $T_Z = T^{**}$.

Accordingly, (Z, T^{**}) is a subspace of (X, T).

Theorem:

Let (Y, T_Y) be a subspace of a topological space (X, T). Let $y \in Y$. Then, a subset M of Y is a $T_Z -$ nhd. of y iff $M = N \cap Y$ for some $T_Z -$ nhd. N of y.

Proof:

Let M be a $T_Y -$ nhd. of y.

Then, $\exists$ a $T_Y -$open set H such that $y \in H \subseteq M$.

Now, H being T_Y-open, we have $H = G \cap Y$ for some $G \in T_Y$.

$y \in G \cap Y \subseteq M$.

Let $\mathbf{M \cup G = N}$.

Then, $\mathbf{y \in G \subset M \cup G = N}$, where $\mathbf{G \in T}$.

This shows that N is a $\mathbf{T - nhd.}$ **of y.**

Further, $\mathbf{N \cap Y = (M \cup G) \cap Y = (M \cap Y) \cup (G \cap Y) = M}$.

$\mathbf{[G \cap Y \subseteq M \text{ and } M \cap Y = M]}$

Conversely, let $\mathbf{M = N \cap Y}$ **for some** $\mathbf{T - nhd.\ N}$ **of y.**

Then, ∃ a T-open set G such that $\mathbf{y \in G \subseteq N}$.

Consequently, $\mathbf{y \in G \cap Y \subseteq N \cap Y = M}$

This shows that M is $\mathbf{T_Y - nhd.}$ **of y.** $\mathbf{[G \cap Y \in T_Y]}$

Sub-Spaces on Topological Space-Continued- The seventh lecture

Theorem:

Let $\mathbf{(Y, T_Y)}$ **be a subspace of a topological space (X, T). Let** $\mathbf{A \subset Y}$**. Then,**

$$\mathbf{cl_Y(A) = cl_X(A) \cap Y.}$$

Proof:

Since $\mathbf{cl_X(A) \cap Y}$ is T-closed, it follows that $\mathbf{cl_X(A) \cap Y}$ is $\mathbf{T_Y}$-closed, Thus, $\mathbf{cl_X(A) \cap Y}$ is a $\mathbf{T_Y}$-closed superset of A.

But, $\mathbf{cl_Y(A)}$ being the smallest $\mathbf{T_Y}$-closed superset of A

$$\mathbf{cl_Y(A) \subseteq cl_X(A) \cap Y} \qquad \ldots\ldots\ldots \quad \text{(i)}$$

Again, $\mathbf{cl_Y(A)}$ being $\mathbf{T_Y}$-closed, we have

$\mathbf{cl_Y(A) = F \cap Y}$ for some T-closed set F.

$\mathbf{A \subseteq cl_Y(A) = F \cap Y}$ **& so** $\mathbf{A \subseteq F}$.

Now, $\mathbf{A \subseteq F \Rightarrow cl_X(A) \subseteq F = F}$ [□ F is T-closed]

$$\mathbf{cl_X(A) \cap Y \subseteq F \cap Y = cl_Y(A)} \qquad \ldots\ldots\ldots \quad \text{(ii)}$$

Hence, from (i) & (ii) we have, $\mathbf{cl_Y(A) = cl_X(A) \cap Y}$.

Theorem:

Let $\mathbf{(Y, T_Y)}$ be a subspace of a topological space (X, T). Let $\mathbf{A \subset Y}$. Then, a point $\mathbf{y \in Y}$ is a T-limit point of A if and only if y is a T-limit point of A.

Proof:

y is a $\mathbf{T_Y}$-limit point of A.

$$\Leftrightarrow [\mathbf{M} - \{\mathbf{y}\}] \cap \mathbf{A} \neq \emptyset \ \forall\ \mathbf{T_Y}\text{- nhds M of y}$$

$$\Leftrightarrow [(\mathbf{N} \cap \mathbf{Y}) - \{\mathbf{y}\}] \cap \mathbf{A} \neq \emptyset \ \forall\ \mathbf{T_Y}\text{- nhd. N of y}$$

$\Leftrightarrow (\mathbf{N}-\{\mathbf{y}\}) \cap \mathbf{A} \neq \emptyset\ \forall\ \mathbf{T_Y}$**- nhd. N of y**

$\Leftrightarrow$ **y is a T-limit point of A.**

Remark: **If $\mathbf{D_Y(A)}$ and $\mathbf{D_X(A)}$ denote the derived sets of A in $(\mathbf{Y}, \mathbf{T_Y})$ & $(\mathbf{X}, \mathbf{T})$ respectively, then**

$\mathbf{D_Y(A) = D_X(A) \cap Y}$.

Theorem:

Let $(\mathbf{Y}, \mathbf{T_Y})$ be a subspace of a topological space (X, T). Let $\mathbf{A \subset Y}$. Then, $\mathbf{T_Y}$-int (A) $\supset$ T-int (A).

Proof:

y $\in$ T- int (A) $\Rightarrow$ y is a T-interior point of A

$\Rightarrow$ **A is a T-nhd. of y**

$\Rightarrow$ **$\mathbf{A \cap Y}$ is a $\mathbf{T_Y}$-nhd. of y**

$\Rightarrow$ **A is a $\mathbf{T_Y}$-nhd. of y** $\quad$ **[$\mathbf{A \subset Y \Rightarrow A \cap Y = A}$]**

$\Rightarrow$ **A $\in$ $\mathbf{T_Y}$-int (A)**

$\mathbf{T_Y}$-int (A) $\supset$ T-int (A)

Remark:

In general, $\mathbf{T_Y}$-int (A) $\neq$ T-int (A).

Example: Let X={ a, b, c, d, e} and let T={ X, ∅, {a}, {a, b}, {a, c, d}, {a, b, e}, {a, b, c, d} be a topology on X.

Let Y={ a, c, e}. Then, T_Y={Y,∅, {a}, {a, c}, {a, e}}.

Now, if A={a, c}⊂ Y, then clearly

T_Y-int (A)={a, c} and T-int (A)={a}.

Thus, in general, T_Y-int (A) ≠ T-int (A).

Theorem:

Let (Y, T_Y) be a subspace of a topological space (X,T) and let A ⊂ Y. Then, T_Y-bd (A) ⊂ T-bd (A).

Proof:

$$y \in T_Y\text{-bd } (A) \Rightarrow y \in cl_Y(A) \cap cl_Y(Y - A)$$

$$\Rightarrow y \in cl_Y(A) \;\&\; y \in cl_Y(Y - A)$$

$$\Rightarrow y \in cl_X(A) \;\cap\; y \in \{cl_X(Y - A)\} \cap Y$$

$$\Rightarrow y \in cl_X(A) \;\&\; y \in cl_X(X - A) \qquad [\overline{Y - A} \subseteq \overline{X - A}\,]$$

$$\Rightarrow y \in T - bd\ (A).$$

T_Y-bd (A) ⊂ T-bd (A)

Theorem:

Let (Y, T_Y) be a subspace of a topological space (X,T). Let β be a base for T, then $\beta_Y = \{B \cap Y : B \in \beta\}$ is a base for T_Y.

Proof:

Let H be any T_Y-open set and let $x \in H$. Then, $\exists$ a T-open set G such that $H = G \cap Y$.

Now, G is a T-open set containing x & β is base for T.

So, $\exists$ a set B in β such that

$$x \in B \subseteq G$$

$$\therefore \quad x \in B \cap Y \subseteq G \cap Y = H$$

Thus, to each $H \in T_Y$ & $x \in H \ \exists \ B \cap Y \in \beta_Y$ such that

$$x \in B \cap Y \subseteq H.$$

This shows that β_Y is a base for T_Y.

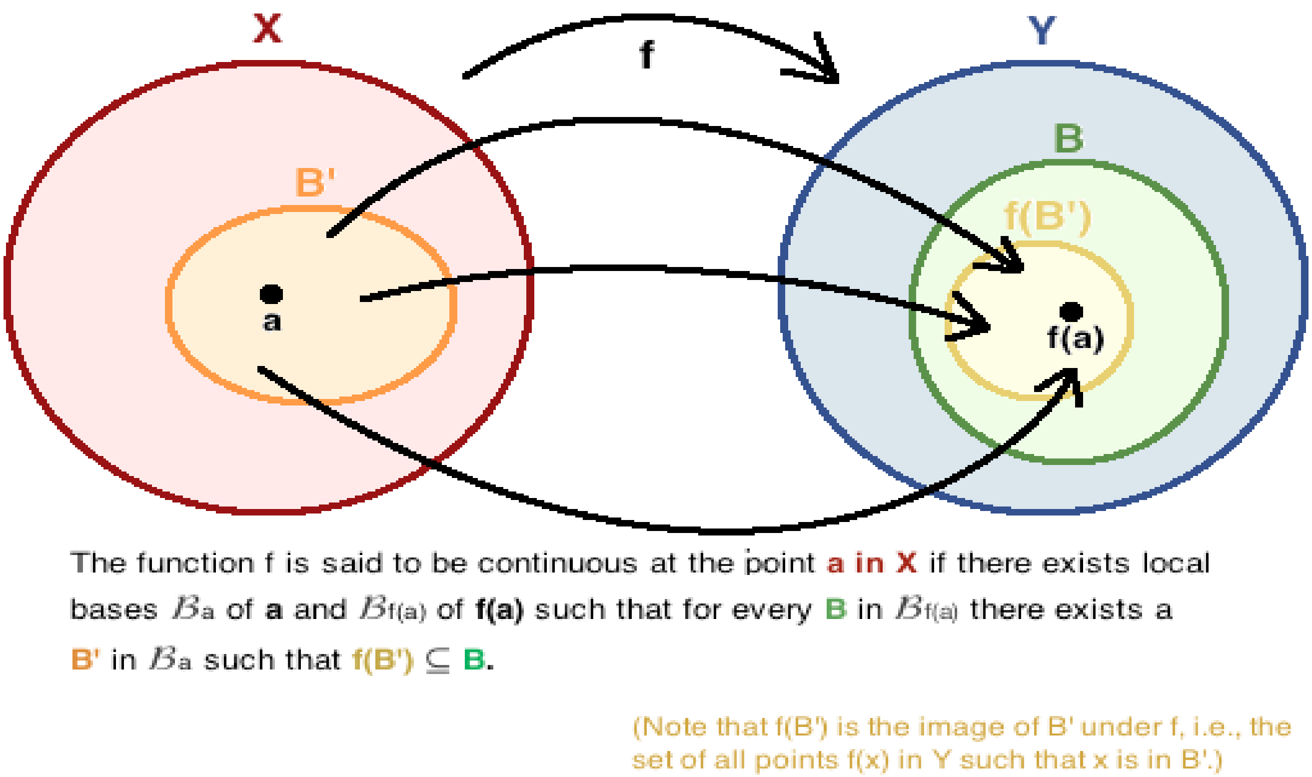
***Definition*: A property of topological space is called or said to be a hereditary property if it is satisfied by every sub spaces of the given space**

The Continuity in Topological spaces- The eighth lecture

ان مفهوم الاستمرارية يبين صنفاً من الدوال ذا اهميه خاصه ليس فقط في دراسة الرياضيات نفسهاً بل حتى في الاستخدامات العديدة في الهندسة والفيزياء حيث ان هذا الصنف من الدوال دوراً مهما , فالاستمرارية من مفاهيم الرياضيات الاساسية ذات المدلول الهندسي المباشر على مخطط الدالة وقولنا ان الدالة مستمرة في نقطة ما يضن ان مخططها في تلك النقطة متصل مع بقيه اجزاءه . وسنقدم في هذا الفصل مفهوم استمراريه الدوال في الفضاءات التبولوجيه بشكل عام وتقدير مبرهنات مهمة توضح هذا المفهوم في هذه الفضاءات . كما تضمن هذا الفصل دراسة موضوع الفضاءات الجزئية او ما تسمى بالفضاءات النسبية ودراسة مفهوم الاستمرارية في هذا الفضاء وتقديم اهم الخواص المتعلقة بهذا الموضوع .

Definition:

Let(X, T) and (Y, T) be topological spaces and Let $f: X \to Y$ be Map Then f is said to continuous at $x \in X$ iff for each U open in Y ($f(x) \in U$) $\exists$ an open set V in x containing x ($x \in V$) such that $f(V) \subseteq U$.



Remark:

If the mapping f continuous at each $x \in X$ then the mapping is called continuous mapping

Example:

Let X = {1, 2, 3} and T = {$\emptyset$, X, {1}, {2}, {1, 2}}

Let Y = {a, b} and T={∅ , Y, {a}} and

f **: X → Y defined as *f* (1)= a ,*f* (2) = *f* (3)=b**

G = X → Y defined g (1) = b , g (2)= g (3) = a

Then *f* is continuous mapping but g is not continues mapping.

Example:

Let *f*: (X, T) → (X, T) be a constant mapping then *f* is continuous.

Proof:-

Let *f*: X→ Y defend by *f* (x) ∈ c , ∀ x ∈. X

Let U be open subset in Y then:

$$f^{-1}(U) = \int_{\emptyset \quad if \;\; C \in U}^{X \quad if \;\; C \in U}$$

Since ∅ and X are open subset then *f* is continuous.

Example:

Let x= { a, b, c, d} and T={∅,X,{a}, {a,b},{b,c,d} , {b}}and *f*: (X,T)→ (X,T) be a mapping by : *f*(a)=   ,*f* (c)=b , *f* (b)=d , *f* (d)=c Then

1- *f* if not continuous .

2- *f* continuous at point d.

3- *f* not continues at point c.

Definition:

A mapping f: (X, T) → (Y, T*) is open mapping iff U is open in x then f (U) is open in Y.

Example:

Let (X,T)be topology space and let Y={a, b, c} and T ={ $\emptyset$, Y, {a}, {a, c}} Then a mapping f:$X \rightarrow Y$ defined as :f (x)= , ∀ x ∈ X is open.

***Definition*:**

A mapping $f: (x, y) \rightarrow$ (Y,T^*) is closed iff E is closed in X then f(E)closed in Y.

Example:

Let (X, T) be a topology and Y={a, b, c}and T={Ø ,Y, {a},{a, c}then mapping $f: x \rightarrow y$ defined $f(x) = b; \forall\, x \in X$ is closed

Results on continuous mapping in Topology spaces- The ninth lecture

Theorem:

Let (X, T) and (Y, T^*) be topology space let $f: x \to y$ then f is continuous iff the in veres image under f of every open set in y is open x.

Proof:-

Let f be continuous and let H be any open satiny if $f^{-1}(H) = \varnothing$, it is clearly open so let $f^{-1}(H) \neq \varnothing$, let $x \in f^{-1}(H)$

Then $f(x) \in H$ Bycontinuity of f, $\exists$ an open set Gin X such that $x \in G$ and $f(G) \subseteq H$. consequently $x \in G \subseteq f^{-1}(H)$

This show that $f^{-1}(H)$ is nhd of each of its points and therefore, it is open in X.

Conversely , let the inverse image under f of every open set in y be open in X, then in order to show that if is continues it is sufficient to show that it is continues at an arbitrary point $x \in X$ let H be any open set in y such that $f(x) \in H$. Then $x \in f^{-1}(H)$ .by hypothesis $f^{-1}(H)$ is an open set in X Now, if we set $f^{-1}(H) = G$, Then h is an open set in X contain x such that $f(G)= f[f^{-1}(H)] \subseteq H$. This Show that f is continues at each point of X.

Theorem:

Let (X, T) and (Y, T) topology space and let *f*: X →Y Then F is continuous iff for each x ∈ X The inverse image under F of every T – nhd of *f* (x) is T-nhd of X.

Proof:-

Let *f* be continuous and let x ∈ X let M be and T – nhd of *f* (x) then ∋ an open set Hiny Such that *f*(x) ∈ H ⊆ f^{-1}(M) Since *f* is continuous and His *T* open , So f^{-1}(H) is T- open this Show that f^{-1}(M) is a T – nhd of X.

Conversely , let the inverse image under f of every T-nhd of f(x) be a T-nhd of X let H be any open set in Y note is f^{-1}(H) = ∅, it is clearly open So let f^{-1}(H) ≠ ∅ and let x ∈ f^{-1}(H) then *f* (x) ∈ H, This show that His a T- nhd of *f* (x). So by hypothesis ; *f* (H) is aT- nhd of X .Thus f^{-1}(H) is a T-nhd of each of its points and there its open, so it follows that in verse image under *f* of every open Sub of Y is an open Sub Set of X. Hence *f* is continuous.

Theorem:

Let (X, T) and (Y,T) be topology space and *f*: X ⟶ Y then f is continuous iff the inverse image under *f* of every closed Subset of Y is a closed sub set of X.

Proof:-

let *f* be continuous and let K be any closed sub set of Y then (Y-k) is an open sub set of y so by continuity of *f*, f^{-1}(Y-k) is an open sub set of X But, f^{-1}(Y^{-1}(k)) is open and therefore f^{-1}(k) is closed

Conversely: - let the inverse image under f of every closed sub set Y then (Y-H) is closed and therefore by hypothesis f^{-1}(Y-H) is closed But f^{-1}(Y-H) = f^{-1}(H) = X-f^{-1}(H). So X. f^{-1}(H) is closed and therefore f^{-1}(H) is open. Thus the inverse image under *f* of every open sub set of Y is an open sub set of X. This show that f is continuous.

Theorem:

Let X, Y and Ƶ be any three Top – Spaces and let *f*: X ⟶ Y and g: Y ⟶ Ƶ be cont mappings Then the composite *gof* : X⟶ Ƶ is continuous.

Proof:-

Let H be any open sub set in Ƶ, we must prove that (*gof*)$^{-1}$(H) is open sub set in X. Since *g* is cont ⟶ g^{-1}(H) is open sub set in Y and Since *f* is cont ⟶ $f^{-1}(g^{-1}(H))$ is open sub set in X So, $f^{-1}(g(H)) = (f^{-1} o g^{-1})(H) = (gof^{-1})(H)$ is open in X. Thus The inverse image under (*gof*) of every open sub set of Ƶ is open sub set of X.

Theorem:

Let (X, T) and (Y,T*) be a Topological space and Let *f*: X→ Y Then *f* is continuous iff for every B ⊆ Y; $f^{-1}(B) \subseteq f^{-1}(\overline{B})$

***Proof:*- Let *f* be continuous and let B ⊆ Y. Then $\overline{B}$ being closed, by continuity of *f*, $f^{-1}(\overline{B})$ is closed $\therefore f^{-1}(B) = f^{-1}(\overline{B})$. Now, $B \subseteq \overline{B} \Rightarrow f^{-1}(B) \subseteq f^{-1}(\overline{B}) \Rightarrow f^{-1}(B) \subseteq f^{-1}(\overline{B})$.**

Conversely:- Let $f^{-1}(B) \subseteq f^{-1}(\overline{B})$ for every B ⊆ Y.

Now, let K be a closed sub set of Y so that $\overline{f}$ = K.

Now, by hypothesis, $f^{-1}(K) \subseteq f^{-1}(\overline{K}) = f^{-1}(K)$. But, $f^{-1}(K) \subseteq f^{-1}(K)$

$\therefore$ $f^{-1}(K) = f^{-1}(K)$ showing that $f^{-1}(K)$ is closed thus the inverse image under f of every closed sub set of Y is a closed sub set of X Hence f is continuous.

Results on continuous mapping in Topology spaces-Continued-

The tenth lecture

Theorem:

Let (X, T) and (Y,T*) be topology space and let $f: X \rightarrow Y$ then f is continuous iff for every $B \subseteq Y$, $\{f^{-1}(B)\}^{\circ} \subseteq f^{-1}(B^{\circ})$

Proof:-

Let f be continues and let $B \subseteq y$ .then B° being open, by continuity of f, $f^{-1}(B^{\circ})$ is consequently $\{ f^{-1}(B^{\circ}) \subseteq f^{-1}(B) \Rightarrow \{ f^{-1}(B^{\circ}) \}^{\circ} \subseteq \{ f^{-1}(B) \}^{\circ}$

$\Rightarrow \{ f^{-1}(B^{\circ}) \subseteq \{ f^{-1}(B) \}^{\circ} \therefore \{ f^{-1}(B)\}^{\circ} \supseteq f^{-1}(B)^{\circ}$ for every $B \subseteq Y$

Conversely:- Let $\{ f^{-1}(B) \}^{\circ} \supseteq f^{-1}(B)^{\circ}$ for every $B \subseteq Y$

Let H be any open sub set of Y , so that $H^{\circ} = H$

$\therefore$ By given hy phthisis $f^{-1}(H) \subseteq \{ f^{-1}(H)\}^{\circ}$ Or

$f^{-1}(H) \} \subseteq \{ f^{-1}(H) \}^{\circ}$ But $\{f^{-1}(H) \}^{\circ} \subseteq \{ f^{-1}(H) \therefore \{f^{-1}(H) \}^{\circ} = f^{-1}(H)$

Showing that f-1(H) is open Thus, the inverse image under *f* of every open sub set of Y is on open subset of X Hence *f* is continuous

Theorem:

Let X, Y and Z be any Three topological spaces Let $f: X \to Y$ and $g: Y \to Z$ be continues mapping then, the composite mapping (gof) :(X→ Z is continuous

Proof:-

Let H be any set open z. then by continuity of g, g^{-1} (H) is open in Y And by continuity of $f^{-1}\{g^{-1}$ (H) } is open in X

So , $f^{-1}\{ g^{-1}(H) \} = (f^{-1} o g^{-1}) (H)$ is open in x. Thus the inverse image under (g of) of every open subset of Z is an open subset of X. Hence of is continuous.

Theorem:

Let (X, T) be a Top – SP and let $Y \subseteq X$ Then the collection:-

$T_y = \{G \cap Y: G \in T\}$ is Topology on X.

Proof:-

(1) $\emptyset \in T$ And $\emptyset \cap Y = \emptyset \Rightarrow \emptyset \in T_y$.

$X \in T$ And $X \cap Y = Y \Rightarrow Y \in T_y$.

(2) let H_1 and H_2 be any tow sets in T_y, we must prove That

$H_1 \cap H_2 \in T_y$.

Since $H_1 \in T_1 \Rightarrow H_1 = G_1 \cap Y$ for some $G_1 \in T$

Since $H_2 \in T_y \Longrightarrow H_2 = G_2 \cap Y$ for some $G_2 \in T$

So , $H_1 \cap H_2 = (G_1 \cap Y) \cap (G_2 \cap Y)$

$= (G_1 \cap G_2) \cap Y \in T_y$

$\therefore H_1 \cap H_2 \in T_y$

(3) let $\{ H_x : \alpha \in \triangle\}$ beany family of setsin T_y.

We must prove that $\cup \{H_{x:}\ \alpha \in \triangle\} \in T_y$.

Since $\{ H_x :_{:}\ \alpha \in \triangle\} \in T_y$. so that for each $_{:}\ \alpha \in \triangle$

$\ni$ A set $G\alpha \in T$ "S – t " $H_x = G \ltimes \cap Y \Longrightarrow \cup \{ H_x : \alpha \in \triangle\} = \cup \{ G_x \cap Y : \alpha \in \triangle\} = [\cup \{G_x : \alpha \in \triangle\}] \cap Y \in T_y$ Therefore $\therefore \cup \{ H_x : \alpha \in \triangle\} \in T_y$.

Theorem:

Let (X, T) and (Y,T^*) be topology space and let f: $X \longrightarrow Y$ be continuous let $A \subseteq X$ Then the restriction f_A of f to A is T_A – T continuous

proof:-

let H be and T^* - open sub set of Y then $f_A^{-1}(H) = A \cap f^{-1}(H)$

Note, by continuity of f , f^{-1} is T- open and therefore:

$A \cap f^{-1}(H)$ is T_A – open. Consanguinity, $f_A^{-1}(H)$ is T_A – open Thus the inverse may under f_A of every T – open sub set of Y is a T_A – open sub set of A.

Theorem:

Let (X, T) and (Y, T^*) be topology space and f: X ⟶ Y be one – one and continuous. Then f maps every dense in itself subset of X on to dense in itself subset of Y

Proof:-

Let A be dense init self subset of X , Then every point of A is a limit point of A . Let Y ∈ f(A). Then f being one- one ∃ aunigue X∈ A such Y = f (x) new let N be T-nhd of f (x) then by continuity of f, f^{-1}(N) is a T – nhd of X But. X∈ A being a limit point of Af^{-1}(N) must contain at least a point Ƶ ≠ X of A.

Now Ƶ ∈ f^{-1}(N) ⟹ f (Ƶ) ∈ (N).

AISO Ƶ ≠ X ⟹ f (Ƶ) ≠ f (X) = Y

Thus N contains at least one point f ((Ƶ) of f (A) defferent fromy. This shows thaty is a limit point of f (A) thus each point of f (A) is a limit point of f (A) . Hence f (A) is dense in itself.

Theorem:

E every continuous image of separable space is separable.

Proof:-

Let (X,T) be separable and let (Y, T^*) be topological space let be a continuous mapping of X on to Y Now X being separable ∃ a countable subset A of X Such that $\overline{A} = X$ ∴ Y = f (X) = f ($\bar{A}$) ⊆ $\overline{f(A)}$

So, $\overline{f(\text{A})} = \text{Y}$. $\therefore f(\text{A}) \subseteq \text{Y}$, always. Also $f(\text{A})$ is countable dense subset of Y, Hence (Y, T^*) is separable.

The homeomorphism in topological Spaces- The eleventh lecture

INTRODUCTION

المقدمـــــة

يعتبر مفهوم التشاكل او التكافؤ التبولوجي من المفاهيم المهمة في هذا الفصل ثم دراسة مفهوما مهما لاقل اهمية عن مفهوم الاستمرارية جدا فيعلم التوبولوجيا وتكمن اهمية هذا المفهوم في كونه أنه بعض الصفات التبولوجية مثل كون المجموعة مفتوحة او مغلقة هي صفات تبولوجية تنذل بفعل التشاكل التبولوجي وذلك كون التشاكل يلعب دورا رئيسيا ومهما في انتقال الصفات التبولوجية من فضاء تبولوجي او فضاء تبلوجي اخر مثل الترابط. وقد تمكنا في هذا الفصل من

دراسة مفهوم التشاكل وأهم خواصه وصفاته التي يتمتع بها الفصل الثالث يتألف من بندين رئيسيين تطرقنا في البند الاول لمفهومي الدوال المفتوحة والمغلقة ودراسة خواصها لدورها البارز بالنسبة لمفهومي الاستمرارية والتشاكل التبولوجي وأهم النتائج المتعلقة بهذا المفهوم التكافؤ التبولوجي بين الفضاءات التبولوجية.

Definition:

Let (X, T) and (Y, T^*) be Top – spaces and let f: X$\longrightarrow$ Y. Then f is said to be:-

1 – Open mapping iff the image under f if every open set in X is open set in Y.

2 – Closed mapping iff the image under f of every closed set in X is closed set in Y.

3 – bi-continuous mapping iff f open and continuous.

Example:

Let (X, T) and (Y, T^*) be a topology spaces: where

$Y = \{a, b, c\}$ And $T^* = \{ \emptyset, Y, \{a\}, \{a, c\}\}$ then a mapping f: X $\longrightarrow$ Y defined as:

$f(x) = a, \forall x \in X$ Is open since for any u is T-open set, we have:

$$f(u) = \begin{cases} \emptyset & \text{When } u = \emptyset \\ \{a\} & \text{When } u \neq \emptyset \end{cases}$$

And each on of $\emptyset$ and {a} is T^* - open set.

$\{f(u)$ is open in y $\forall$ u open Subset in X}

Example:

Let (X, T) and (Y, T^*) be topology spaces and let $Y = \{a, b, c\}$ and $T^* = \{\emptyset, Y, \{a\}, \{a, c\}\}$

Then the mapping f: X $\longrightarrow$ Y defined as:- $f(x) = b, \forall x \in X,$ is closed mapping since for any f is T – closed set ,

$$f(F) = \begin{cases} \emptyset & \text{When } F = \emptyset \\ \{b\} & \text{When } F \neq \emptyset \end{cases}$$

And each one of $\emptyset$ and {b} is T^* - closed

{f (F) is closed in Y $\forall$ F closed sub set in X.}

Theorem:

Let (X, T) and (Y, T^*) be topology space let f:X $\longrightarrow$ Y Then f isopen iff $f(\mathring{A}) \subseteq [f(A)]^\circ$ for every $A \subset X$

Proof:

Let f be open, Then A° being open it follows that f (A)° is open consequently, [f (A)°]° = f (A°)

Now: $A^\circ \subseteq A \Rightarrow f(A)^\circ \subseteq f(A)$

$$\Rightarrow [f(A)^\circ \subseteq [f(A)]^\circ$$

$$\Rightarrow f(A)^\circ \subseteq [f(A)]^\circ$$

Conversely: let f (A°) ⊆ $[f(A)]^\circ$ for every A ⊂X

Let A be an open subset of X so that A° = A

$\therefore\ f(A)^\circ \subseteq [f(A)]^\circ \Rightarrow f\ (A) \subseteq [f(f)]^\circ \ \therefore [A^\circ = A$ **But** $[f(A)]^\circ \subseteq f\ (A)$

$\therefore [f(A)]^\circ = f\ (A)$. **This show that f (A) is open when every A is open.**

Theorem:

Let (X,T) and (Y, T)* be topology space let f: X → Y then f is closed iff $\overline{f(A)} \subseteq f\left(\overline{A}\right)$ for every A ⊂X.

proof: **Let f be closed and let A ⊂X. then $\overline{A}$ being closed $f\left(\overline{A}\right)$ is therefore closed consequently $\overline{f(\bar{A})} = f(\bar{A})$**

Now, $A \subseteq \bar{A} \Rightarrow f\ (A) \subseteq f\ (\bar{A})$

$$\Rightarrow \overline{f(A)} \subseteq \{f(\bar{A})\}$$

Hence, $\overline{f(\mathbf{A})} \subseteq f\,(\bar{\mathrm{A}})$ for very A⊂X

Let A be a closed subset of X then A = A^{-}

$\therefore \overline{f(\mathbf{A})} \subseteq f\,(\bar{\mathrm{A}}) \Rightarrow \overline{f(\mathbf{A})} \subseteq f(\mathbf{A})\ [\therefore\ \bar{\mathrm{A}} = \mathbf{A}]$

But, $f(\mathbf{A}) \subseteq f\,(\mathbf{A})$ Therefore $\overline{f(\mathbf{A})} = f(\mathbf{A})$.

This show that *f* (A) is closed, when every so is A. Hence *f* is a closed mapping.

Definition:

Two topological spaces (X, T) and (Y, T*) are closed homeomorphic if there exits: One – to – one and onto function *f*: X ⟶ y such that *f* and *f* $^{-1}$ are continuous and the function f is called homeomorphism.

Example:

Let $\mathbf{X} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ and $\mathbf{T} = \{\emptyset, \mathbf{X}, \{\mathbf{a}\}, \{\mathbf{b}\}, \{\mathbf{a}, \mathbf{b}\}\}$

And $\mathbf{Y} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ and T* $\{\emptyset, \mathbf{y}, \{\mathbf{c}\}, \{\mathbf{d}\}, \{\mathbf{c}, \mathbf{d}\}\}$

And *f*: X ⟶ Y defined as:

***f* (a) = a , *f* (b) = b , *f* (c) = c, *f* (d) = d. is (X, T)**

And (Y, T*) are homeomorphic?

1 – *f* is one – to – one and onto But *f* is not continuous since {c} ∈ T* But

***f* $^{-1}${c} = {c} ∉ T . Therefore f is not homeomorphic.**

Example:

Let $X = \{a, b, c, d\}$; $T = \{\emptyset, y, \{c\}, \{d\}, \{c, d\}\}$ and $g: (X, T) \to (Y, T)^*$ such that:-

g (a) = d , g (b) = c , g (c)= b , g(d) = a. is

(X, T) and (Y, T^*) are homeomorphic?

Sol:-

(1) And (2) are clear g is one – to – one and onto.

(3) Is g continuous?

(*) $Y \in T^* \to g^{-1}(y) = X \in T$.

(*) $g^{-1}\{\emptyset\} = \emptyset \in T$

(*) $g^{-1}\{c\} = \{b\} \in T^*$, $g^{-1}\{d\} = \{a\} \in T$ and

(*) $g^{-1}\{c, d\} = \{a, b\} \in T$. So g is continuous,

(4) is g^{-1} continuous?

(*) $(g^{-1})^{-1}\{a\} = g\{a\} = \{d\} \in T^*$

(*) $(g^{-1})^{-1}\{\emptyset\} = \emptyset \in T^*$

(*) $(g^{-1})^{-1}\{b\} = g\{b\} = \{c\} \in T$

(*) $(g^{-1})^{-1}\{X\} = Y \in T^*$.

(*) $(g^{-1})^{-1}\{a, b\} = g\{a, b\} = \{d, c\} \in T^*$

Since g is one – to – one, onto, g and g^{-1} are continuous.

So, g is homeomorphism. Therefore (X, T) and (Y, T^*) are homeomorphic

Theorem:

let (X,T) and (Y, T^*) be topology space let *f* be a one – one mapping of on to Y then the following statements are all equivalent to one another:-

(i) *f* is open continuous.

(ii) *f* is homeomorphism .

(iii) *f* is closed and continuous.

Proof:- **(i) ⟹ (ii) let *f* be a one – one open and continuous mapping of X onto Y then by definition it is a homeomorphism so (i) ⟹ (ii)**

(ii) ⟹ (iii) : let *f* be homeomorphism. Then it is a one – one continuous open mapping of X onto Y. Let *f* beany closed subset of X then (X-*f*) is open

Now *f* being open it follows that *f* (X-*f*) is open But

f (X-f) = f (X) – f (F) = Y-f (F)

Thus Y-*f* (F) is open and therefor, *f* (F) is closed. This show that *f* is closed and continuous so (ii) ⟹ (iii)

(iii) ⟹ (i) : let *f* be closed and continuous let G be an open subset of X then X – G is closed and being closed f (X-G) is therefore, closed

But, f (X-G) = f (X) – f (G) = Y-f (G)

Thus, Y. *f* (G) is closed and therefore, *f* (G) is open this show that *f* is closed and continuous.

So (iii) $\Rightarrow$ (i) Thus , (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) Hence all the given statements are equivalent to on another

Theorem:

Let (X, T) and (Y, T^*) be topology space let $f : X \longrightarrow Y$ be a one – one mapping of X on to y then f is a homeomorphism iff $f(\bar{A})$ for every $A \subset X$

Proof:

Let f be homeomorphism. Then f is a one – one continuous and closed mapping of X onto Y.

Let $A \subset X$ then by continuity of f, we have $f(\bar{A}) \subseteq f(A)$

Also, f being closed we have $f(\bar{A}) \subseteq f(A)$, hence $f(\bar{A}) = f(\bar{A})$

Conversely: - let $f : X \longrightarrow y$ such that is f is one – one onto and for every $A \subset X$, let $f(\bar{A}) = \overline{f(A)}$ Then $f(\bar{A}) \subseteq f(\bar{A})$ and $\overline{f(A)} \subseteq f(\bar{A})$. But these results show that f is continuous and closed f is one – one onto also, so it a homeomorphism.

The homeomorphism in topological Spaces-Continued- The twelfth lecture

Theorem:

Let (X, T) and (Y, T^*) be topology space let f: $X \longrightarrow Y$ Be a one – one mapping of X onto Y then f is a homeomorphism iff $f\{A\}^\circ = \{f(A)\}^\circ$ for every $A \subset X$.

Proof:

Let f be homeomorphism. Then f is are one-one continuous and open mapping of X onto Y let $A \subseteq X$.

Then f being open we have $f(A^\circ) \subseteq \{f(A\}^\circ \ldots\ldots..(1)$

Also f being continuous and on to and $f(A) \subset Y$ we have $f^{-1}[(f(A)]^\circ = A^\circ$

$[f^{-1}(B^\circ) \subseteq \{f^{-1}(A)]^\circ$ for every $B \subset y$

Or $[f(A)^\circ \subseteq f(A)^\circ \ldots\ldots..(2)$

Thus for (1) and (2) we get $f(A)^\circ = [f(A)]^\circ$

Conversely:-

Let f be one-one mapping of X onto y such that $f(A)^\circ = [f(A)]^\circ$ for every $A \subset X$ Then $f(A)^\circ \subseteq [f(A)]^\circ$ and $[f(A)]^\circ \subseteq f(A)^\circ$

But $f(A)^\circ \subseteq [f(A)]^\circ$ for every $A \subset X$ implies that f is open agin let $B \subset X$. Such that $B = f(A)$ or $A = f^{-1}(B)$

Now,

$$[f(A)]^\circ \Rightarrow f(A^\circ) \Rightarrow f^{-1}[f(A)^\circ] \subseteq [f^{-1}[f(A^\circ)] = A^\circ$$

$$\Rightarrow f^{-1}(B^\circ) \subseteq [f^{-1}(B)]^\circ$$

$\therefore \{A = f^{-1}(B)$ and $f(A) = B\}$

is homeomorphism

$$\therefore (X, T) \approx (Y, T^*) \Rightarrow (Y, T^*) \approx (X, T)$$

(iii) Transitivity:-

Let $(X,T) \approx (Y,T^*)$ and $(Y,T^*) \approx (Ƶ,T^*)^*$ and let f and g be the corresponding homeomorphisms Then f is one-one onto, T^*-continuous and f^{-1} is T^*-T continuous Also g is one-one onto T^*-T^{**}continuous and g^{-1} is T^{**}-T^* continuous we claim that the composite mapping *gof*: $X \longrightarrow Ƶ$ is a homeomorphisms since the composite of tow continuous mapping beings being continuous it follows that got is T,T^{**} continuous more-over g^{-1} is T^{**},T^* continuous and f^{-1} is T^*-T continuous.

$\Rightarrow f^{-1} o g^{-1}$ is T^{**}-T continuous

$\Rightarrow (gof)^{-1}$ is T^{**}-T continuous. Thus *gof* is homeomorphisms and then $(X,T) \approx (Ƶ, T^{**})$.

Hence the relation of homeomorphism on the set of all topology space is an equivalence relation. This shows that $f^{-1}(B^\circ) \subseteq \{f^{-1}(B)\}^\circ$ for every $B \subset y$ so f is continuous. Thus, f is a one – one continuous open mapping of X on to Y Hence f is a homeomorphism.

Theorem:

The relation of homeomorphism on the set of all topological spaces is equivalence.

Proof:-

This relation satisfies the following properties:

(i)-Reflexivity let (X, T) be any topology space then the identity mapping I: X $\longrightarrow$ X: $I(X) = X$.

Is clearly one-one onto, I is continuous for if G $\in$ T, Then I^{-1}(G)= G $\in$ T .

Also I is open, for if G $\in$ T , Then I(G) = G $\in$ T .

Thus I is homeomorphism. $\boldsymbol{Therefore}\ (\boldsymbol{X},\boldsymbol{T}) \approx (\boldsymbol{X},\boldsymbol{T})$

(ii) Symmetry: - let $(\boldsymbol{X},\boldsymbol{T}) \approx (\boldsymbol{y},\boldsymbol{T})^{*}$ and let f be the corresponding homeomorphism. Then f is one-one onto T-T^* continuous and open. Now f is one-one onto $\Longrightarrow f^{-1}$ is one-one onto f is open $\Longrightarrow f^{-1}$ is T-T^* continuous $\Longrightarrow (f^{-1})^{-1}$ is T-T^* continuous Thus show that the mapping f^{-1} : Y $\longrightarrow$ X.

Connectedness and Compactness in Topological Spaces

The thirteenth lecture

SEPARATION AXIOMS (T_0, T_1, T_2, $T_{2\frac{1}{2}}$ T_3,$T_{3\frac{1}{2}}$, T_4, AND T_5)

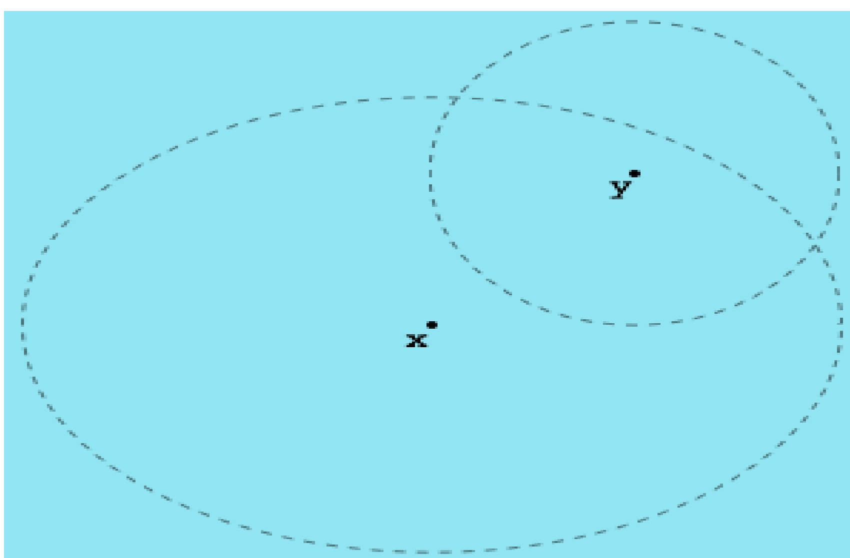
AND RELATIONSHIPS AMONG THEM

T_0-property and spaces

A topological space X has the T_0-property if there exists an open set which separates any two distinct points: if *x* and *y* are distinct points of X, there exist an open set which contains one but not the other. Let me be more explicit. A topological space X has the T_0 property if, for any two distinct points x and y in X, either there exists an open set M(x) containing x which does not contain y, or there exists an open set N(y) containing y which does not contain x.

NOTE: that the space X is an open set containing x, but it contains y, and vice versa.

Here's a picture of T_0, showing an open set containing y that does not contain x. A T_0 space is sometimes, but rarely do I think, called Kolmogorov.



The fourteenth lecture

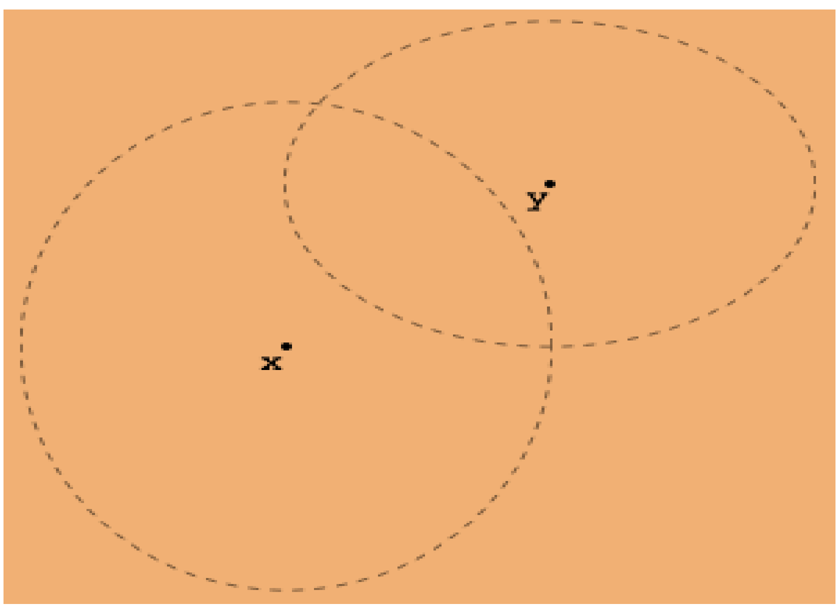
T_0-Space and T_1 property and spaces

A topological space X has the T_1 property if x and y are distinct points of X, there exists an open set M(x) which contains x but not y, and an open set N(y) which contains y but not x.

One crucial property of a T_1 space is that points (singleton sets) are closed.

This time each point has an open set which contains it but not the other.

NOTE: that we did not assert that the two open sets do not intersect, merely that their intersection contains neither x nor y. (That's the next property.) Here's a picture of T_1, showing open sets which intersect, but their intersection, as we require, does not contain x or y. A T_1 space is sometimes, but again rarely, I think, called Frechet.

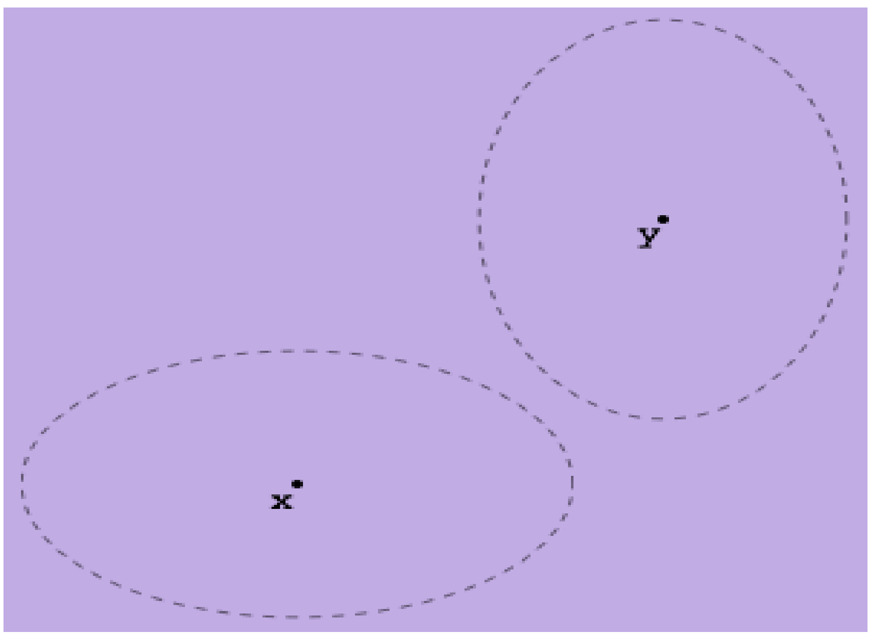


T_1-spaces

The fifteenth lecture

T_2 property and spaces

A topological space X has the T_2 property if x and y are distinct points of X, there exist <u>disjoint</u> open sets M(x) and N(y) containing x and y respectively. Here's a picture of T_2. A T_2 space is almost always, in my experience, called Hausdorff. One crucial property of a Hausdorff space is that limit points are unique. (No, I haven't defined a limit point. That's another interesting subject.)



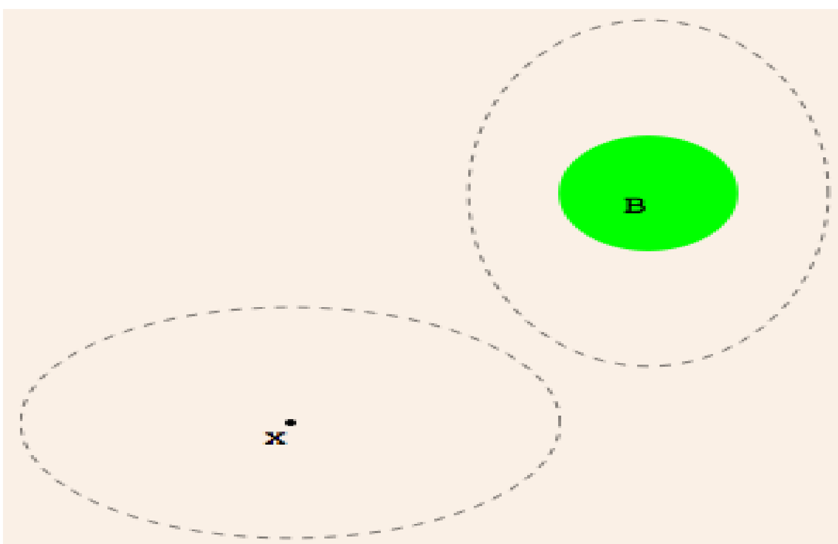
T_2-spaces

The sixteenth lecture

T_3 and regular

Now we look at separating <u>sets</u> instead of points, still separating them by open sets of some kind. First we separate a point and a closed set. (A set A in X is closed if its complement X – A is open; the closure of A ($\bar{A}$), is the smallest closed set containing A.) A topological space X has the T_3 property if there exist disjoint open sets which contain any closed set and any point not in the set: for any closed set B and any point $x \notin B$, there exist disjoint open sets containing x and B respectively.

Here's T_3. This time I use uppercase ("*B*") and color to denote the closed set.



T_3-spaces

It is crucial that the following set and topology (shown earlier as "an intermediate example") is T_3 but not T_1 (the problem is that the point is not closed):

X = {a, b, c} . $T = \{\emptyset, X, \{a\}, \{b, c\}\}$

This is why and where we need to combine properties in order to get especially worthwhile topological spaces. (Yes, we can study T_3, T_4, and T_5 spaces per se. it is more fruitful to study $T_3 + T_1$, $T_4 + T_1$, and $T_5 + T_1$)

We say that a space is regular if it is T_1 and T_3.

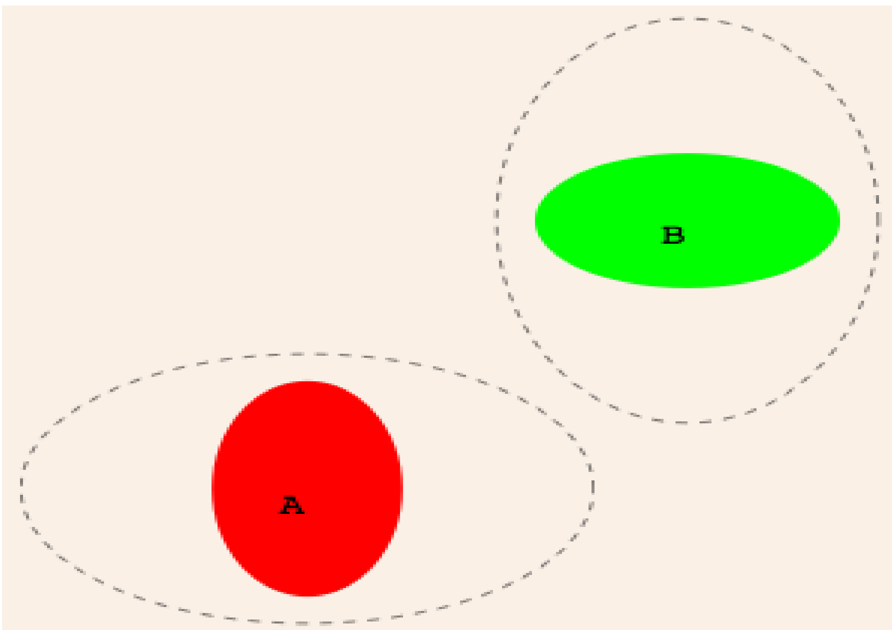
(In fact, we can show that if a space is T_0 and T_3, then it is T_2, hence T_1, hence T_1 and T_3. this means we could have defined a space as regular if it is T_0 and T_3. Of course, T_1 and T_3 immediately implies T_0 and T_3, so the two possible definitions of "regular" are equivalent.)

Although I used "normal" and "T_4" in the introductory discussion, the alternative terminology appears here as well, It applies to all subscripts 3 and higher. Where I say that a topological space is regular iff it is T_1 and T_3 other people use regular to refer to my T_3 property, and say a topological space is T_3 iff T_1 and regular. Whereas the progression of the earlier separation axioms kept tightening the requirements on the open sets whose existence we asserted, here we just replaced a point by a closed set. That would be a refinement of the earlier property if points themselves were closed sets. But that's T_1, and that's why we want to study spaces which are both T_1 and T_3.

The seventeenth lecture

T_4 and normal

Now we separate two closed sets instead of a point and a closed set. A topological space X has the T_4 property if there exist disjoint open sets which contain any two disjoint closed sets: for any disjoint closed sets A and B, there exist disjoint open sets containing A and B respectively.



T_4-spaces

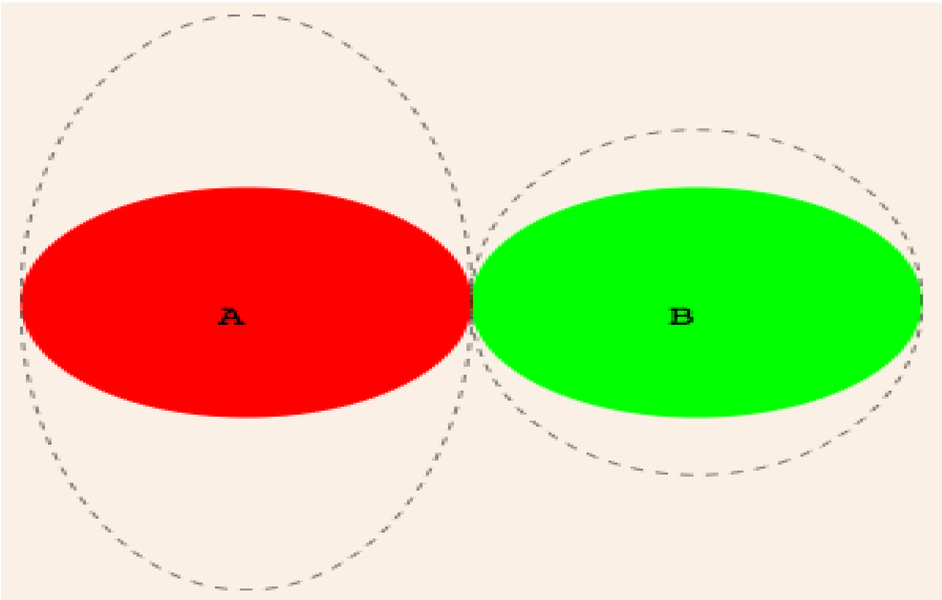
I should mention that a bad property of T_4 spaces is that T_4 is not hereditary: not every subspace of T_4 is T_4. We say that a space is normal if it is T_1 and T_4. We still have the analogous: not very subspace of a normal space is normal.

The eighteenth lecture

T_5 and completely normal

Two subsets A and B of topological space are separated if $\bar{A} \cap B = \emptyset = A \cap \bar{B}$.

A topological space X has the T_5 property if there exist disjoint open sets which contain any two separated sets: for any separated sets A and B, there exist disjoint open sets containing A and B respectively.



T_5-spaces

Should mention that an alternative equivalent definition of T_5 is that: a space is T_5 iff every subspace is T_4. It corrects the problem with T_4.

We say that a space is completely normal if it is T_5 and T_1. We have the analogous: a space is completely normal iff every subspace is normal. It corrects the problem with normal, too.

Consider the two open intervals A = (0, 1/2) and B = (1/2, 1) with the usual topology of the real line. The sets do not intersect: $A \cap B = \emptyset$, but the closed intervals, their closures, do: $\bar{A} = [0, \frac{1}{2}]$

$\bar{B}$ = [$\frac{1}{2}$, 1] and $\bar{A} \cap \bar{B} = \{\frac{1}{2}\}$, Nevertheless, A and B are separated, because $\bar{A} \cap B = \emptyset = A \cap \bar{B}$.

A and B have the T_5 property because A and B themselves are disjoint open sets. All of those properties, T_0 thru T_5, asserted the existence of open sets, sometimes satisfying additional conditions.

The nineteenth lecture

$3\frac{1}{2}$ *And Completely Regular*

We have an intermediate property which is described differently.

Given two disjoint subsets A and B of a space X, a Urysohn function for A and B is a continuous function $f: X \longrightarrow [0, 1]$ such that f (A) = 0 and f (B) = 1.

Urysohn's Lemma, then, says that if A and B are disjoint closed subsets of a T_4 space, then there exists a Urysohn function for A and B.

A topological space X has the $3\frac{1}{2}$ property if there exist a real-valued continuous function which separates an open set from any point not in it: (i.e.) for each open set $U \subset X$ and each x not in U, there exist a Urysohn function f for x and U.

We say that a space is completely regular (or Tychonoff) if it is $3\frac{1}{2}$ and T_1.

The twenty lecture

Implications of the properties

At this point, thanks to adding T_1 to the definitions, we can show (!)

Completely normal ⇒ normal ⇒ completely regular ⇒ regular ⇒ $T_{2\frac{1}{2}}$ ⇒ T_2 ⇒ T_1 ⇒ T_0

The implications among the T_i properties (for i > $2\frac{1}{2}$) are not so pretty.

Note that a Urysohn space was not in that list. Instead of the subsequence completely regular ⇒ regular ⇒ $T_{2\frac{1}{2}}$

We could have written completely regular ⇒ Urysohn ⇒ $T_{2\frac{1}{2}}$.

But there is no inclusion relationship between Urysohn and regular. We have two beautiful inclusions, if we omit either regular or Urysohn, but not if we include both.

This is the second reason why I decided to follow Steen & Seebach and use T's for the properties and names for the combinations. If we did it the other way, with names for the properties and T's for the combinations, we could write

T_5 ⇒ T_4 ⇒ $T_{3\frac{1}{2}}$ ⇒ T_3 ⇒ $T_{2\frac{1}{2}}$ ⇒ T_2 ⇒ T_1 ⇒ T_0, or, more elegantly,

Ti ⇒ Tj for i > j, with i, j in {0, 1, 2, $T_{2\frac{1}{2}}$, 3, $T_{3\frac{1}{2}}$, 4, 5}

But then we've left Urysohn spaces out in the cold. Since the theorem is no longer pretty, I chose to use the shorter Ti to denote a property, and write, for example, normal = T_1 + T_4.

I first saw them the other way: T_4 = normal + T_1, etc.

And it is possible that I would not have been so struck by them without the lovely $Ti \Rightarrow Tj$ for, $i > j$. (Adamson emphasizes that he chooses this convention because of the simplicity of that statement.) Nevertheless, I have presented them the other way. The fact is, if you're studying someone else's work, you may have to adopt their terminology as long as you're there.